Decreasing the maximum degree of a graph

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Abstract

For a non-empty graph G, let $\lambda(G)$ be the smallest number of vertices that can be deleted from G so that the maximum degree of the resulting graph is smaller than the maximum degree $\Delta(G)$ of G. If G is regular, then $\lambda(G)$ is the domination number $\gamma(G)$ of G. We show that if $1 \leq k < r$ and c is a real number such that $\gamma(H) \leq c|V(H)|$ for every connected k-regular graph H with $|V(H)| \geq r$, then $\lambda(G) \leq c|V(G)|$ for every connected graph G with $\Delta(G) = k$ and $|V(G)| \geq r$. We in fact show that

$$\lambda(G) \leq \frac{\gamma(H)}{|V(H)|}|V(G)|$$

for an H explicitly constructed from G. Several bounds on $\lambda(G)$ follow. Various problems motivated by the result are posed, and related results are obtained.

We also provide a sharp bound on $\lambda(G)$ that depends only on the vertices of largest degree. We call a vertex of largest degree a Δ-*vertex*. We call a Δ-vertex v *solitary* if no other Δ-vertex is of distance at most 2 from v. Let $S(G)$ be the set of solitary Δ-vertices, and let $T(G)$ be the set of non-solitary Δ-vertices. We show that

$$\lambda(G) \leq |S(G)| + \frac{\Delta(G)}{\Delta(G)+1}|T(G)|.$$

The bound can be attained with $T(G) \neq \emptyset$.

1 Introduction

Unless stated otherwise, we use small letters such as x to denote non-negative integers or elements of sets, and capital letters such as X to denote sets or graphs. The set of positive integers is denoted by $\mathbb{N}$. For $n \geq 1$, $[n]$ denotes the set $\{1, \dots, n\}$ (that is, $[n] = \{i \in \mathbb{N} \colon i \leq n\}$). We take $[0]$ to be the empty set $\emptyset$. Arbitrary sets are assumed to be finite. For a set X, $\binom{X}{2}$ denotes the set of 2-element subsets of X (that is, $\binom{X}{2} = \{\{x, y\} \colon x, y \in X, x \neq y\}$).

If Y is a subset of $\binom{X}{2}$ and G is the pair (X, Y), then G is called a *graph*, X is called the *vertex set of* G and denoted by $V(G)$, and Y is called the *edge set of* G and denoted by $E(G)$. Arbitary graphs are assumed to have non-empty vertex sets. A *vertex of* G is an element of $V(G)$, and an *edge of* G is an element of $E(G)$. We call G an n*-vertex graph* if $|V(G)| = n$. We may represent an edge $\{v, w\}$ by vw. If $vw \in E(G)$, then we say that w is a *neighbour of* v *in* G (and vice-versa). For $v \in V(G)$, $N_G(v)$ denotes the set of neighbours of v in G, $N_G[v]$ denotes $N_G(v) \cup \{v\}$, and $d_G(v)$ denotes $|N_G(v)|$ and is called the *degree of* v *in* G. The *minimum degree of* G is $\min\{d_G(v) \colon v \in V(G)\}$ and is denoted by $\delta(G)$. The *maximum degree of* G is $\max\{d_G(v) \colon v \in V(G)\}$ and is denoted by $\Delta(G)$. If $v \in V(G)$ and $d_G(v) = \Delta(G)$, then we call v a *max-degree vertex of* G or a Δ*-vertex of* G. The set of Δ-vertices of G is denoted by $M(G)$. For $X \subseteq V(G)$, $N_G[X]$ denotes $\bigcup_{v \in X} N_G[v]$ (the *closed neighbourhood of* X), $G[X]$ denotes $(X, E(G) \cap \binom{X}{2})$ (the *subgraph of* G *induced by* X), and $G - X$ denotes $G[V(G) \backslash X]$ (the graph obtained by *deleting* X from G).

A *copy* of a graph H is a graph obtained by relabeling the vertices of H. More formally, if $\phi : V(H) \to V(G)$ is a bijection and $E(G) = \{\phi(v)\phi(w) \colon vw \in E(H)\}$, then G is said to be a *copy of* H or *isomorphic to* H, and we write $G \simeq H$.

For $n \geq 1$, the graphs $([n], \binom{[n]}{2})$ and $([n], \{\{i, i+1\} \colon i \in [n-1]\})$ are denoted by K_n and P_n, respectively. For $n \geq 3$, C_n denotes the graph $([n], \{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}, \{n, 1\}\})$ $(= ([n], E(P_n) \cup \{n, 1\}))$. A copy of K_n is called a *complete graph*. A copy of P_n is called an n*-path* or simply a *path*. A copy of C_n is called an n*-cycle* or simply a *cycle*. A graph G is called k*-regular*, or simply *regular*, if $d_G(v) = k$ for each $v \in V(G)$.

If G and H are graphs such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is called a *subgraph of* G, and we say that G *contains* H.

If $G, G_1, \dots, G_t$ are graphs such that $V(G) = \bigcup_{i=1}^{t} V(G_i)$ and $E(G) = \bigcup_{i=1}^{t} E(G_i)$, then G is the *union of* $G_1, \dots, G_t$. If $V(G_i) \cap V(G_j) = \emptyset$ for every $i, j \in [t]$ with $i \neq j$, then $G_1, \dots, G_t$ are *pairwise vertex-disjoint*. A graph G is *connected* if, for every $v, w \in V(G)$, G contains a path P with $v, w \in V(P)$. A connected subgraph H of G is a *component of* G if, for each connected subgraph K of G with $K \neq H$, H is not a subgraph of K. Clearly, any two distinct components of G are pairwise vertex-disjoint.

We call a subset D of $V(G)$ a Δ*-reducing set of* G if no vertex of $G - D$ has degree $\Delta(G)$, that is, if $\Delta(G - D) < \Delta(G)$ or $\Delta(G) = 0$ and $D = V(G)$. Note that D is a Δ-reducing set of G if and only if $M(G) \subseteq N_G[D]$. Let $\lambda(G)$ denote the size of a smallest Δ-reducing set of G. We call $\lambda(G)$ the Δ*-reducing number of* G.

For $X, D \subseteq V(G)$, we say that D *dominates* X *in* G if $X \subseteq N_G[D]$. Thus, D dominates X if and only if, for each $v \in X$, v is in D or has at least one neighbour in D. Note that D dominates $M(G)$ in G if and only if D is a Δ-reducing set of G. Thus, $\lambda(G) = \min\{|D| \colon D \text{ dominates } M(G) \text{ in } G\}$. A *dominating set of* G is a set that dominates $V(G)$ in G. The size of a smallest dominating set of G is called the *domination number of* G and denoted by $\gamma(G)$. Thus, the problem of minimizing the size of a Δ-reducing set is a variant of the classical domination problem [3, 5, 9, 10, 11, 12]; the aim is to use as few vertices as possible to dominate the vertices of maximum degree rather than all the vertices. Many other variants have been studied (see, for example, [4, 6, 8, 13, 14]); many of the earliest ones are referenced in [12], but nowadays there are several others. If G is k-regular, then our problem is the same as the classical one, that is, $\lambda(G) = \gamma(G)$.

The parameter $\lambda(G)$ was introduced and studied in [1], and the extremal structures for bounds in [1] were determined in [2]. An application is indicated in [25]. In this paper, we provide new upper bounds on $\lambda(G)$, establishing, in particular, connections with domination numbers of regular graphs.

2 Results

For $v \in V(G)$, we denote $N_G[N_G[v]]$ (the set of vertices of G of *distance* at most 2 from v) by $N_G^{(2)}[v]$, and we denote $N_G^{(2)}[v]\backslash\{v\}$ by $N_G^{(2)}(v)$. We call a Δ-vertex v of G *solitary* if $N_G^{(2)}(v) \cap M(G) = \emptyset$. Let $S(G)$ denote the set of solitary Δ-vertices of G, and let $T(G)$ denote $M(G)\backslash S(G)$ (the set of non-solitary Δ-vertices of G).

The following is our first main result, proved in Section 3.

Theorem 1 *For any graph G,*

$$\lambda(G) \leq |S(G)| + \frac{\Delta(G)}{\Delta(G)+1}|T(G)|.$$

Moreover, for $k \geq 0$, the bound is attained by infinitely many non-isomorphic connected graphs G with $\Delta(G) = k$ and $T(G) \neq \emptyset$ if and only if $k \geq 3$.

The bound was conjectured by Yair Caro, Kurt Fenech, and the author (see [7]). If $\Delta(G) = 0$, then $T(G) = \emptyset$. If G is a connected graph with $\Delta(G) = 1$, then $G \simeq K_2$, $T(G) = V(G)$, and the bound is attained. Since $\lambda(G)$ is an integer, Theorem 1 gives us $\lambda(G) \leq |S(G)| + \left\lfloor \frac{\Delta(G)}{\Delta(G)+1}|T(G)| \right\rfloor$. Clearly, the cycles and the paths with at least 3 vertices are the only connected graphs with maximum degree 2, and, for $n \geq 3$, we have $\lambda(P_n) = \left\lfloor \frac{n}{3} \right\rfloor$ (clearly, $\{3i \colon 1 \leq 3i \leq n\}$ is a smallest Δ-reducing set of P_n) and $\lambda(C_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$ (clearly, $\{1+3i \colon 1 \leq 1+3i \leq n\}$ is a smallest dominating set of C_n). Thus, for a connected graph G with $\Delta(G) = 2$ and $T(G) \neq \emptyset$, $\lambda(G) = |S(G)| + \left\lfloor \frac{\Delta(G)}{\Delta(G)+1}|T(G)| \right\rfloor$ if and only if G is a 4-path or a 6-path or a 4-cycle.

We next study the relationship between Δ-reducing numbers and domination numbers. By definition, $\lambda(G) \leq \gamma(G)$, and if G is regular, then $\lambda(G) = \gamma(G)$. Our aim is to bound from above the ratio $\lambda(G)/|V(G)|$ by $c_{\mathcal{R}} = \max\{\gamma(H)/|V(H)| \colon H \in \mathcal{R}\}$ for some set $\mathcal{R}$ of $\Delta(G)$-regular graphs, where $\mathcal{R}$ is as small as possible so that $c_{\mathcal{R}}$ is as small as possible. We show that this is achieved in particular by taking $\mathcal{R}$ to be the set of $\Delta(G)$-regular pn-vertex graphs with $p = 2\left\lceil \frac{\Delta(G)-\delta(G)+1}{2} \right\rceil$ and $n = |V(G)|$. Moreover, from G, we explicitly construct a $\Delta(G)$-regular graph H such that

$$\lambda(G) \leq \frac{\gamma(H)}{|V(H)|}|V(G)| \tag{1}$$

and that, if G is connected, then H is connected. If G is regular, then we take $H = G$ (and the result is immediate). If G is not regular, then H is a pn-vertex graph (see Construction 1 and Lemmas 1 and 2).

If $\alpha(n,k)$ is a real number defined for some $n,k \in \mathbb{N}$, then let

$$\alpha'(n,k) = \frac{\alpha(n,k)}{n}.$$

If G is a k-regular n-vertex graph, then kn is even by the handshaking lemma ($\sum_{v \in V(G)} d_G(v) = 2|E(G)|$). Consider any two integers n and k such that $2 \leq k < n$ and kn is even. The graph $B_{n,k}$ in Section 4 is a connected k-regular n-vertex graph. Let $f(n,k)$ be the smallest rational number c such that the domination number of every connected k-regular n-vertex graph is at most cn. Thus, $0 < f(n,k) < 1$. Let

$$\gamma_{\rm cr}(n,k) = \max\{\gamma(G) \colon G \text{ is a connected } k\text{-regular graph with } V(G) = [n]\}$$

(the subscript 'cr' refers to connected regular graphs). Thus,

$$f(n,k) = {\gamma_{\rm cr}}'(n,k).$$

Together with (1), the following is our second main result, proved in Section 4; part (i) follows immediately from (1).

Theorem 2 (i) *If G is a connected non-regular n-vertex graph with maximum degree $k \geq 2$ and minimum degree ℓ, then*

$$\lambda(G) \leq f\left(2\left\lceil \frac{k-\ell+1}{2} \right\rceil n, k\right) n.$$

(ii) *If, moreover, $\sum_{v \in V(G)} d_G(v) \neq kn - 1$, then, for any even integer $p > k - \ell$,*

$$\lambda(G) \leq f(pn,k)n.$$

Remark 1 (i) If G is connected and k-regular, then $\lambda(G) = \gamma(G) \leq f(n,k)n$.

(ii) Theorem 2 may not hold if G is regular. Indeed, if $G = C_n$ with $n = 3t+1$ for some $t \geq 1$, then $\ell = k = 2$, $2\left\lceil \frac{k-\ell+1}{2} \right\rceil = 2$, and, for any even $p > 0$, $\lambda(G) = \gamma(G) = 1 + t > \frac{1+pt+\lfloor (p-1)/3 \rfloor}{p} = \frac{\gamma(C_{pn})}{p} = \frac{\gamma(C_{pn})}{pn}n = f(pn,2)n$ as every connected 2-regular graph is a cycle.

(iii) The integer $2\left\lceil \frac{k-\ell+1}{2} \right\rceil$ is the smallest even $p > k - \ell$.

(iv) The bounds in Theorem 2 are attained if $G = P_n$ with $n = 3t$ for some $t \geq 1$. Indeed, in such a case, we have $\ell = 1$, $k = 2$, $2\left\lceil \frac{k-\ell+1}{2} \right\rceil = 2$, and, for any even $p > 0$, $\lambda(G) = t = \frac{\gamma(C_{pn})}{pn}n = f(pn,2)n$.

One of the questions we pose in Section 5 is whether the bound on $\lambda(G)$ in Theorem 2 (ii) also holds when $\sum_{v \in V(G)} d_G(v) = kn - 1$ (see Problem 2). We conjecture an affirmative answer (Conjecture 1) and make some observations regarding this case.

We now turn our attention to the desired consequences of Theorem 2. We show how Theorem 2 combines with known domination results to give us explicit values that bound $\lambda(G)$ from above. We start with our main consequence of Theorem 2.

Theorem 3 *If $1 \leq k < r$ and c is a real number such that $\gamma(H) \leq c|V(H)|$ for every connected k-regular graph H with $|V(H)| \geq r$, then $\lambda(G) \leq c|V(G)|$ for every connected graph G with $\Delta(G) = k$ and $|V(G)| \geq r$.*

Proof. Let $\ell = \delta(G)$. Since G is connected, $\ell \geq 1$. If G is k-regular, then $\lambda(G) = \gamma(G) \leq c|V(G)|$. Suppose that G is not k-regular. Then, $\ell < k$, and hence $k \geq 2$. Let $n = |V(G)|$. Let $p = 2\left\lceil \frac{k-\ell+1}{2} \right\rceil$. Since $p > k - \ell \geq 1$ and $n \geq r$, $pn > r$. Thus, $\gamma_{\rm cr}(pn, k) \leq c(pn)$. By Theorem 2 (i), $\lambda(G) \leq \frac{\gamma_{\rm cr}(pn,k)}{pn} n \leq cn$. $\Box$

We call a connected graph G *exceptional* if $\delta(G) = 2$ and $\gamma(G) > \frac{2}{5}|V(G)|$. The exceptional graphs were determined in [20]; they number to 7 (up to isomorphism). The efforts of several authors established that if $k \geq 1$ and G is a connected n-vertex graph with $\delta(G) \geq k$, then, unless $k = 2$ and G is exceptional,

$$\gamma(G) \leq \frac{k}{3k-1} n. \tag{2}$$

The cases $k = 1$, $k = 2$, and $k = 3$ were proved by Ore [21], McCuaig and Shepherd [20], and Reed [22], respectively. Caro and Roditty obtained $\gamma(G) \leq \left(1 - \frac{k}{k+1}\left(\frac{1}{k+1}\right)^{1/k}\right) n$ (see [9]), which is better than (2) for $k \geq 7$. It was then conjectured in [9] that (2) also holds for the remaining cases $k = 4$, $k = 5$, and $k = 6$, which were eventually settled in [23], [24], and [15], respectively. Of the 7 exceptional graphs (see [20, 9]), only C_4 and C_7 are regular. Using these facts, we obtain the following.

Theorem 4 *If G is a connected n-vertex graph with maximum degree k, then, unless G is a copy of C_4 or of C_7,*

$$\lambda(G) \leq \min\left\{\frac{k}{3k-1}, 1 - \frac{k}{k+1}\mu^{1/k}\right\} n,$$

where $\mu = \min\left\{1, \frac{n}{(k+1)|M(G)|}\right\}$.

Proof. If either $k \neq 2$, or $k = 2$ and $n \geq 8$, then $\lambda(G) \leq \frac{k}{3k-1}n$ by (2) and Theorem 3. Clearly, the paths and the cycles are the only connected graphs with maximum degree at most 2. Thus, it is easy to check that if $k = 2$ and $n \leq 7$, then $\lambda(G) \leq \frac{2}{5}n = \frac{k}{3k-1}n$ unless G is a copy of C_4 or of C_7. Let $t = |M(G)|$. Since $M(G)$ is a Δ-reducing set of G, $\lambda(G) \leq t$. If $t \leq \frac{n}{k+1}$, then $\mu = 1$ and $\lambda(G) \leq \frac{n}{k+1} = \left(1 - \frac{k}{k+1}\mu^{1/k}\right)n$. Suppose $t \geq \frac{n}{k+1}$. Then, $\mu = \frac{n}{(k+1)t}$. It is shown in [1, Proof of Theorem 2.7] that $\lambda(G) \leq n\rho + t(1-\rho)^{k+1}$ for any real number ρ satisfying $0 \leq \rho \leq 1$. Using differentiation, we find that the minimum value of $n\rho + t(1-\rho)^{k+1}$ is attained when $\rho = 1 - \left(\frac{n}{(k+1)t}\right)^{1/k}$ (note that this satisfies $0 \leq \rho \leq 1$ as $t \geq \frac{n}{k+1}$), so the minimum value is $\left(1 - \frac{k}{k+1}\mu^{1/k}\right)n$. $\Box$

Note that the bound $\left(1 - \frac{k}{k+1}\mu^{1/k}\right) n$ on $\lambda(G)$ is at most the Caro–Roditty bound on $\gamma(G)$, so it is better than the bound $\frac{k}{3k-1}n$ on $\lambda(G)$ for $k \geq 7$.

Improving a bound in [18], Kostochka and Stocker [19] proved that if $n \geq 9$ and G is a connected 3-regular n-vertex graph, then $\gamma(G) \leq \frac{5}{14}n$. Thus, by Theorem 3 with $k = 3$ and $r = 9$, we immediately obtain the following generalization.

Theorem 5 *If $n \geq 9$ and G is a connected n-vertex graph with maximum degree 3, then*

$$\lambda(G) \leq \frac{5}{14}n.$$

A result similar to Theorem 2 holds if we drop the condition that G is connected. For $1 \leq k < n$ with kn even, let $g(n,k)$ be the smallest rational number c such that the domination number of every k-regular n-vertex graph is at most cn, and let

$$\gamma_{\mathrm{r}}(n,k) = \max\{\gamma(G) \colon G \text{ is a } k\text{-regular graph with } V(G) = [n]\}.$$

Thus,

$$g(n,k) = {\gamma_{\mathrm{r}}}'(n,k).$$

In Section 4, we show that the proof of Theorem 2 also yields the following.

Theorem 6 *If G is an n-vertex graph with maximum degree k and minimum degree ℓ, then, for any even integer $p > k - \ell$,*

$$\lambda(G) \leq g(pn,k)n.$$

Remark 2 (i) If G is k-regular, then $\lambda(G) = \gamma(G) \leq g(n,k)n$.

(ii) The bound in Theorem 6 is attained if $n = 4t + 3$ for some $t \geq 0$, G is an n-vertex graph that is the union of t 4-cycles and a 3-path (so G is non-regular), and $p = 2$. Indeed, let H be a $2n$-vertex graph that is the union of $2t$ 4-cycles and two 3-cycles. We have that $\Delta(G) = 2$, $\delta(G) = 1$, H is 2-regular, and $\gamma(H) = 4t + 2 = 2\lambda(G)$. Let H' be a 2-regular $2n$-vertex graph. Since $2n = 4(2t+1) + 2$, at least one component of H' is not a 4-cycle. Note that $\gamma(C) = \left\lceil \frac{|V(C)|}{3} \right\rceil \leq \frac{|V(C)|}{2} - \frac{1}{2}$ for any cycle $C \not\simeq C_4$, and that $\gamma(C_4) = 2 = \frac{|V(C_4)|}{2}$. Since every 2-regular graph is a cycle or the union of pairwise vertex-disjoint cycles, it follows that $\gamma(H') \leq \frac{|V(H')|}{2} - \frac{1}{2} = 4t + 3 - \frac{1}{2}$, so $\gamma(H') \leq 4t + 2$. Therefore, $\gamma_{\mathrm{r}}(2n,2) = \gamma(H) = 4t + 2$, and hence $\lambda(G) = g(2n,2)n$.

(iii) The bound may be attained if G is regular. This occurs if, in (ii), the 3-path is replaced by a 3-cycle. We also give an example for $k = 3$. Suppose that G is the union of $s \geq 1$ pairwise vertex-disjoint copies of the graph C_8' below. Then, G is 3-regular and $\lambda(G) = \gamma(G) = 3s = \frac{3}{8}n$. By (3) (below), for any even $p > 0$, $g(pn,3) = \frac{3}{8}$, so $\lambda(G) = g(pn,3)n$.

The next result follows from Theorem 6 similarly to the way Theorem 3 follows from Theorem 2.

Theorem 7 *If $0 \leq k < r$ and c is a real number such that $\gamma(H) \leq c|V(H)|$ for every k-regular graph H with $|V(H)| \geq r$, then $\lambda(G) \leq c|V(G)|$ for every graph G with $\Delta(G) = k$ and $|V(G)| \geq r$.*

However, if $G_1, \dots, G_t$ are the components of a graph G, and $\tau(G)$ is $\lambda(G)$ or $\gamma(G)$, then $\tau(G) = \sum_{i=1}^{t} \tau(G_i)$, so, to a large extent, we only need to consider connected graphs. Moreover, the value of $f(n,k)$ is obviously either the same as that of $g(n,k)$ or better than it, that is, $f(n,k) \le g(n,k)$. There are infinitely many integers n for which $f(n,3) < g(n,3)$. In particular,

$$\text{if } n = 8t \text{ for some } t \ge 2, \text{ then } f(n,3) \le \tfrac{5}{14} < \tfrac{3}{8} = g(n,3). \tag{3}$$

Indeed, we have the following. By the above-mentioned Kostochka–Stocker (KS) bound, $f(n,3) \le \frac{5}{14}$. Now it is well-known that there exist 3-regular 8-vertex graphs whose domination number is 3; the graph C_8' with $V(C_8') = V(C_8) = [8]$ and $E(C_8') = E(C_8) \cup \{\{1,5\},\{2,6\},\{3,7\},\{4,8\}\}$ is an example. If G is the union of t pairwise vertex-disjoint copies of C_8', then G is 3-regular and $\gamma(G) = 3t = \frac{3}{8}n$. By (2) with $k = 3$ (due to Reed [22]), we obtain the equality $g(n,3) = \frac{3}{8}$ in (3). Not much more than the results above is known about the values $f(n,k)$ and $g(n,k)$.

If X is a set, a and b are functions with domain X, $a(x) \le b(x)$ for every $x \in X$ (or $a(x) \ge b(x)$ for every $x \in X$), X has an infinite subset Y such that $a(x) = b(x)$ for every $x \in Y$, and either Y is a set of non-isomorphic graphs or Y is not a set of graphs, then we will say that the bound b on a is *infinitely attainable*.

Establishing infinitely attainable domination bounds for connected graphs is a central problem in domination theory and particularly challenging for regular graphs. The bound (2) is infinitely attainable for $k = 1$ [21] and for $k = 2$ [20] (see [9]); however, to the best of the author's knowledge, no bound of the form $c_k|V(G)|$ for some $k \ge 3$ and for connected graphs G with $\delta(G) \ge k$ and $|V(G)| \ge r$ is known to be infinitely attainable. It is also not known if the integer part of the KS bound is infinitely attainable; however, it is at least nearly so, as shown in [16, 17, 18, 19].

The next two sections are dedicated to the proofs of Theorems 1, 2, and 6. In Section 5, we pose several problems and conjectures arising from Theorems 2 and 6, or concerning the associated values defined above and related ones, and we provide some partial answers.

3 Proof of Theorem 1

In this section, we prove Theorem 1.

Proof of Theorem 1. We first prove the bound. Let $n = |V(G)|$ and $k = \Delta(G)$. If $n = 1$, then $\lambda(G) = n = |S(G)|$. We now consider $n > 1$ and proceed by induction on n.

If $k = 0$, then $S(G) = V(G)$, and hence $\lambda(G) = |S(G)|$. Suppose $k \ge 1$.

Suppose $\lambda(G) = 1$. If $S(G) \ne \emptyset$, then $\lambda(G) \le |S(G)|$. By the definition of $T(G)$, $|T(G)| \ge 2$ if $T(G) \ne \emptyset$. Since $M(G) = S(G) \cup T(G)$, $T(G) \ne \emptyset$ if $S(G) = \emptyset$. Thus, if $S(G) = \emptyset$, then $|T(G)| \ge 2$, and hence we have $\lambda(G) = 1 \le \frac{1}{2}|T(G)| \le \frac{k}{k+1}|T(G)|$.

Now suppose $\lambda(G) > 1$. Then, for each $v \in V(G)$, we have

$$\Delta(G - v) = k, \tag{4}$$

and hence

$$M(G - v) \subseteq M(G) = M(G - v) \cup (N_G[v] \cap M(G)). \tag{5}$$

Thus, if $v \in V(G)$ and D is a smallest set that dominates $M(G-v)$ in $G-v$, then $D \cup \{v\}$ is a set that dominates $M(G)$ in G, and hence

$$\lambda(G) \leq \lambda(G-v) + 1. \tag{6}$$

Suppose $S(G) \neq \emptyset$. Let $x \in S(G)$. Since $\Delta(G-x) = k$, we have $S(G-x) = S(G) \backslash \{x\}$ and $T(G-x) = T(G)$. By the induction hypothesis,

$$\lambda(G-x) \leq |S(G-x)| + \frac{k}{k+1}|T(G-x)| = |S(G)| - 1 + \frac{k}{k+1}|T(G)|.$$

By (6), the result follows.

Now suppose $S(G) = \emptyset$. Then, $M(G) = T(G) \neq \emptyset$. We have one of the following two cases.

Case 1: There exist $x, x' \in M(G)$ such that $xx' \in E(G)$. Let $G' = G - x$. Let $Y = \{v \in N_G(x) \colon d_G(v) = k\}$ and $Z = N_G(x) \backslash Y$. Thus, $x' \in Y$ and, by (4), $(Y \cup Z) \cap M(G') = \emptyset$. For each $y \in Y$, let $A_y = S(G') \cap N_{G'}^{(2)}(y)$. Let $A = \bigcup_{y \in Y} A_y$. Let $B = S(G') \backslash A$. Since $M(G') = S(G') \cup T(G') = A \cup B \cup T(G')$, it follows by (5) that $M(G) = A \cup B \cup T(G') \cup \{x\} \cup Y$, and hence

$$|M(G)| = |A| + |B| + |T(G')| + 1 + |Y|.$$

Consider any $y \in Y$. For any $v \in V(G')$, we have $w \in N_{G'}^{(2)}[u]$ for every $u, w \in N_{G'}[v]$, and hence $S(G')$ does not have more than one element in $N_{G'}[v]$. We have

$$|A_y| = \left| S(G') \cap \bigcup_{v \in N_{G'}(y)} (N_{G'}[v] \backslash \{y\}) \right| \leq \sum_{v \in N_{G'}(y)} |S(G') \cap N_{G'}[v]| \leq \sum_{v \in N_{G'}(y)} 1 = |N_{G'}(y)|.$$

Thus, since $x \in N_G(y)$ and $x \notin N_{G'}(y)$, we have $|A_y| \leq k-1$. Therefore, $|A| \leq (k-1)|Y|$.

Suppose $B = \emptyset$. Then, $S(G') = A$. By (6) and the induction hypothesis,

$$\begin{aligned}
\lambda(G) &\leq 1 + \lambda(G') \leq 1 + |A| + \frac{k}{k+1}|T(G')| \\
&= 1 + |A| + \frac{k}{k+1}(|M(G)| - |A| - 1 - |Y|) \\
&= \frac{k|M(G)| + |A| + 1 - k|Y|}{k+1} \leq \frac{k|M(G)| + (k-1)|Y| + 1 - k|Y|}{k+1} \\
&= \frac{k|T(G)| - |Y| + 1}{k+1} \leq \frac{k}{k+1}|T(G)|.
\end{aligned}$$

Now suppose $B \neq \emptyset$.

Consider any $b \in B$. By (4), $d_{G'}(b) = k$, so $b \notin N_G(x)$. Let $X_b = N_G^{(2)}(b) \cap M(G)$. Since $b \in M(G') \subseteq M(G) = T(G)$, $X_b \neq \emptyset$. Let $b' \in X_b$. Since $b \in S(G')$, we have $b' \notin M(G')$ or $b' \notin N_{G'}^{(2)}(b)$. Suppose $b' \in N_{G'}^{(2)}(b)$. Then, $b' \notin M(G')$, and hence $b' \in Y$. This gives us $b \in A$, which contradicts $b \in B$. Thus, $b' \notin N_{G'}^{(2)}(b)$. Since $b' \in N_G^{(2)}(b)$, it follows that $b' = x$ or $x \in N_G(b)$ (and $b' \in N_G(x) \backslash N_G(b)$). Since $b \notin N_G(x)$, $b' = x$. We have therefore shown that $X_b = \{x\}$ (as b' is an arbitrary element of X_b). Since

$x = b' \in N_G^{(2)}(b)$ and $x \notin N_G(b)$, there exists some $z_b \in N_G(b)$ such that $z_b \in N_G(x)$. Since $b \notin A$, we have $z_b \in Z$ and $N_G(z_b) \cap Y = \emptyset$.

Let $Z_B = \{z_b \colon b \in B\}$, $N_G(Z_B) = N_G[Z_B] \backslash Z_B$, and $N_G^{(2)}(B) = \left(\bigcup_{b \in B} N_G^{(2)}(b)\right) \backslash B$. We have shown that $N_G^{(2)}(B) \cap M(G) = \{x\}$, $Z_B \subseteq Z$, and

$$N_G(Z_B) \cap Y = \emptyset. \tag{7}$$

Since $Z_B \subseteq Z$, we have

$$Z_B \cap M(G) = \emptyset \quad \text{and} \quad Z_B \cap M(G') = \emptyset. \tag{8}$$

Suppose $v^* \in M(G)$ for some $v^* \in N_G(Z_B) \backslash (B \cup \{x\})$. Then, $v^* \notin Z$ (as $d_G(v^*) = k$), $v^* \notin Y$ (by (7)), and hence $v^* \notin N_G(x)$. Thus, $d_{G'}(v^*) = k$, and hence $v^* \in M(G')$. Since $v^* \in N_G(Z_B)$, $v^* \in N_{G'}(z_{b^*})$ for some $b^* \in B$. We obtain $v^* \in N_{G'}^{(2)}(b^*)$ ($v^* \neq b^*$ as $v^* \notin B$ by assumption), which contradicts $b^* \in S(G')$.

Therefore,

$$N_G(Z_B) \cap M(G) = B \cup \{x\}.$$

Suppose $\Delta(G - Z_B) < k$. Then, $M(G) = B \cup \{x\}$. Recall that $M(G) = T(G)$. We have

$$\lambda(G) \leq |Z_B| \leq \frac{|Z_B|}{|Z_B|+1}(|B|+1) < \frac{|Z \cup Y|}{|Z \cup Y|+1}|T(G)| = \frac{k}{k+1}|T(G)|.$$

Now suppose $\Delta(G - Z_B) = k$. Then,

$$M(G) = M(G - Z_B) \cup (N_G(Z_B) \cap M(G)) = S(G - Z_B) \cup T(G - Z_B) \cup B \cup \{x\} \tag{9}$$

and

$$M(G - Z_B) \cap (B \cup \{x\}) = \emptyset. \tag{10}$$

Subcase 1.1: $S(G - Z_B) = \emptyset$. Let D be a smallest subset of $V(G - Z_B)$ that dominates $M(G - Z_B)$ in $G - Z_B$. By the induction hypothesis, $|D| \leq \frac{k}{k+1}|T(G - Z_B)|$. By (9), $D \cup Z_B$ dominates $M(G)$ in G. Thus,

$$\begin{aligned} \lambda(G) &\leq |D| + |Z_B| \leq \frac{k}{k+1}|T(G - Z_B)| + \frac{|Z_B|}{|Z_B|+1}(|B|+1) \\ &< \frac{k}{k+1}|T(G - Z_B)| + \frac{|Z \cup Y|}{|Z \cup Y|+1}(|B|+1) = \frac{k}{k+1}(|T(G - Z_B)| + |B \cup \{x\}|), \end{aligned}$$

and hence $\lambda(G) < \frac{k}{k+1}|M(G)|$ by (9) and (10). Recall that $M(G) = T(G)$. Thus, $\lambda(G) < \frac{k}{k+1}|T(G)|$.

Subcase 1.2: $S(G - Z_B) \neq \emptyset$. Let $u \in S(G - Z_B)$. By (9), $u \in M(G)$. Since $d_{G-Z_B}(u) = k = d_G(u)$, we have $u \notin N_G(Z_B)$, so $u \notin B \cup \{x\}$.

Suppose $u \notin N_G[x]$. Then, $u \notin \{x\} \cup Y \cup Z$. If we assume that $u \in N_G(y)$ for some $y \in Y$, then, by (7), we obtain $u, y \in T(G - Z_B)$, which contradicts $u \in S(G - Z_B)$. Thus, $u \notin N_G[Y]$. Since $u \notin N_G[x]$, $d_{G'}(u) = k$. Thus, $u \in S(G')$ or $u \in T(G')$. Suppose

$u \in S(G')$. Since $u \notin B$, $u \in A$. Thus, $u \in N_G^{(2)}(y^*)$ for some $y^* \in Y$. Since $u \notin N_G[Y]$, we have $uv', v'y^* \in E(G)$ for some $v' \in V(G)\backslash\{u, y^*\}$. By (7), $y^* \notin N_G(Z_B)$, so $v' \notin Z_B$. Since $u, y^* \notin N_G[Z_B]$, we obtain $u \in N_{G-Z_B}^{(2)}(y^*)$ and $u, y^* \in T(G-Z_B)$, which contradicts $u \in S(G-Z_B)$. Thus, $u \in T(G')$, and hence there exists some $u' \in V(G)\backslash N_G[x]$ such that $d_{G'}(u') = k$ and $u' \in N_{G'}^{(2)}(u)$. Suppose $N_G(u') \cap Z_B \neq \emptyset$. Then, $z_{b'} \in N_G(u')$ for some $b' \in B$. Since $b' \in S(G')$ and $u \in N_{G'}^{(2)}(u')$, $u' \neq b'$. Since $u', b' \in N_{G'}(z_{b'})$, we obtain $u' \in N_{G'}^{(2)}(b')$, which contradicts $b' \in S(G')$. Thus, $N_G(u') \cap Z_B = \emptyset$. Since $u \notin N_G(Z_B)$, $N_G(u) \cap Z_B = \emptyset$. Since $u' \in N_G^{(2)}(u)$, we obtain $u' \in N_{G-Z_B}^{(2)}(u)$ and $d_{G-Z_B}(u') = k = d_{G-Z_B}(u)$, contradicting $u \in S(G-Z_B)$.

Therefore, $u \in N_G[x]$. Since $u \notin N_G(Z_B)$, $u \neq x$. Thus, since $u \in N_G(x) = Y \cup Z$ and $d_G(u) = k$, $u \in Y$. We have therefore shown that $S(G-Z_B) \subseteq Y$ (as u is an arbitrary element of $S(G-Z_B)$). Suppose $|Y| \geq 2$. Let $y^* \in Y\backslash\{u\}$. By (7), $y^* \in M(G-Z_B)$. Since $u, y^* \in N_{G-Z_B}(x)$, we obtain $y^* \in N_{G-Z_B}^{(2)}(u)$, which contradicts $u \in S(G-Z_B)$. Thus, $|Y| = 1$, and hence

$$Y = \{u\} = S(G-Z_B). \tag{11}$$

Let D be a smallest subset of $V(G-Z_B)$ that dominates $M(G-Z_B)$ in $G-Z_B$. By the induction hypothesis, $|D| \leq 1 + \frac{k}{k+1}|T(G-Z_B)|$. By (9), $D \cup Z_B$ dominates $M(G)$ in G. By (9) and (10), we have $M(G) = \{u\} \cup T(G-Z_B) \cup (B \cup \{x\})$, and the sets $\{u\}$, $T(G-Z_B)$, and $B \cup \{x\}$ are pairwise disjoint. Thus, $|M(G)| = |T(G-Z_B)| + |B| + 2$. Recall that $M(G) = T(G)$. We have

$$\begin{aligned}
\lambda(G) &\leq |D| + |Z_B| \leq 1 + \frac{k}{k+1}|T(G-Z_B)| + |Z_B| \\
&\leq \frac{k}{k+1}|T(G-Z_B)| + \frac{|Z_B|+1}{|Z_B|+2}(|B|+2) \\
&\leq \frac{k}{k+1}|T(G-Z_B)| + \frac{|Z \cup Y|}{|Z \cup Y|+1}(|B|+2) \\
&= \frac{k}{k+1}|T(G-Z_B)| + \frac{k}{k+1}(|B|+2) = \frac{k}{k+1}|M(G)| = \frac{k}{k+1}|T(G)|.
\end{aligned}$$

Case 2: No two vertices in $M(G)$ are neighbours in G, that is,

$$v \notin N_G(w) \text{ for every } v, w \in M(G). \tag{12}$$

Let $x \in M(G)$. By (4), $\Delta(G-x) = k$.

Suppose $S(G-x) \neq \emptyset$. Then, we can apply the argument in Case 1; in the present case, $Y = \emptyset$ by (12), and hence $B = S(G-x) \neq \emptyset$. The strict inequalities for $\lambda(G)$ arising from $|Z_B| < |Z \cup Y|$ in Case 1 now become non-strict (as Y is now empty). The argument in Sub-case 1.2 shows us that Sub-case 1.2 now does not arise (that is, we do not have $S(G-Z_B) \neq \emptyset$), because (11) gives $Y \neq \emptyset$.

Now suppose $S(G-x) = \emptyset$. Since $M(G) = T(G)$, there exists some $x' \in N_G^{(2)}(x)$ such that $x' \in M(G)$. Thus, by (12), $xu, ux' \in E(G)$ for some $u \in V(G)\backslash M(G)$. Let $Y_u = \{v \in N_G(u) \colon d_G(v) = k\}$ and $Z_u = N_G(u)\backslash Y_u$. Then, $x, x' \in Y_u$, so $|Y_u| \geq 2$. Let

$G' = G - u$. By (4), $\Delta(G') = k$. For each $y \in Y_u$, let $A_y = S(G') \cap N_{G'}^{(2)}(y)$. As in Case 1, $|A_y| \leq k-1$ for each $y \in Y_u$.

Suppose $S(G') \neq \bigcup_{y \in Y_u \backslash \{x\}} A_y$. Then, there exists some $v \in S(G')$ such that

$$v \notin A_y \text{ for each } y \in Y_u \backslash \{x\}. \tag{13}$$

Since $d_{G'}(v) = k$, we have $v \notin N_G[u]$ and $v \in M(G)$. By (12), $x \notin N_G(v)$, so $v \in M(G-x)$. Since $S(G-x) = \emptyset$, $v \in T(G-x)$. Thus, there exists some $w \in N_{G-x}^{(2)}(v)$ such that $w \in M(G-x)$. By (12), $w \notin N_{G-x}(v)$, so $vv', v'w \in E(G-x)$ for some $v' \in V(G-x)\backslash\{v,w\}$. Since $v \notin N_G[u]$, $v' \neq u$. Also, $w \neq u$ as $d_{G-x}(w) = k = d_G(w)$ and $d_G(u) < k$. Thus, $vv', v'w \in E(G')$, and hence $w \in N_{G'}^{(2)}(v)$. Since $v \in S(G')$, we consequently have $w \notin M(G')$. Thus, since $d_G(w) = k$, we have $w \in N_G(u)$, and hence $w \in Y_u\backslash\{x\}$. Now $v \in N_{G'}^{(2)}(w)$ as $w \in N_{G'}^{(2)}(v)$. Thus, we have $v \in A_w$, which contradicts (13).

Therefore, $S(G') = \bigcup_{y \in Y_u \backslash \{x\}} A_y$, and hence

$$|S(G')| \leq \sum_{y \in Y_u \backslash \{x\}} |A_y| \leq \sum_{y \in Y_u \backslash \{x\}} (k-1) = (k-1)(|Y_u|-1).$$

Let $s = |S(G')|$. Since $\Delta(G') = k$, we have $M(G) = M(G') \cup (M(G) \cap N_G[u]) = M(G') \cup Y_u$ and $M(G') \cap Y_u = \emptyset$, so $|M(G)| = |M(G')| + |Y_u|$. By the induction hypothesis,

$$\begin{aligned}\lambda(G') &\leq s + \frac{k}{k+1}(|M(G')| - s) = \frac{s + k|M(G')|}{k+1} \\ &= \frac{s + k(|M(G)| - |Y_u|)}{k+1} = \frac{s + k(|T(G)| - |Y_u|)}{k+1},\end{aligned}$$

and hence, by (6),

$$\begin{aligned}\lambda(G) &\leq 1 + \lambda(G') \leq \frac{k+1+s+k(|T(G)|-|Y_u|)}{k+1} = \frac{k|T(G)| - k(|Y_u|-1) + s + 1}{k+1} \\ &\leq \frac{k|T(G)| - k(|Y_u|-1) + (k-1)(|Y_u|-1) + 1}{k+1} \leq \frac{k}{k+1}|T(G)|\end{aligned}$$

as $|Y_u| \geq 2$.

We have proved the bound in the theorem. We now prove the second part of the theorem. The case $k \leq 2$ was settled in Section 2. Suppose $k \geq 3$. If F is a graph with $|V(F)| = p+1$ and $E(F) = \{vw \colon w \in V(F)\backslash\{v\}\}$ for some $v \in V(F)$, then F is a *p-star* with *center* v. Loosely speaking, we join the non-center vertices of r k-stars to the centers of rk $(k-1)$-stars in a one-to-one way, and we add some edges to the resulting graph so that we obtain a graph $I_{r,k}$ that is connected and attains the bound in the theorem. More precisely, let $J_{r,k}$ be the union of pairwise vertex-disjoint graphs $F_1, \dots, F_r, H_{1,1}, \dots, H_{1,k}, \dots, H_{r,1}, \dots, H_{r,k}$, where $F_1, \dots, F_r$ are k-stars and $H_{1,1}, \dots, H_{1,k}, \dots, H_{r,1}, \dots, H_{r,k}$ are $(k-1)$-stars. For each $i \in [r]$, let $u_{i,0}$ be the center of F_i, let $u_{i,1}, \dots, u_{i,k}$ be the members of $V(F_i)\backslash\{u_{i,0}\}$, let $x_i, x_i' \in V(H_{i,1})$ with $x_i \neq x_i'$ and $d_{H_{i,1}}(x_i) = d_{H_{i,1}}(x_i') = 1$ (x_i and x_i' exist as $k-1 \geq 2$), and, for each

$j \in [k]$, let $w_{i,j}$ be the center of $H_{i,j}$. Let $I_{r,k}$ be the graph with $V(I_{r,k}) = V(J_{r,k})$ and $E(I_{r,k}) = E(J_{r,k}) \cup \{u_{i,j}w_{i,j} : i \in [r],\, j \in [k]\} \cup \{x_i'x_{i+1} : i \in [r-1]\}$. Suppose $G = I_{r,k}$. Let $U = \{u_{i,0} : i \in [r]\}$, $U' = (\bigcup_{i=1}^{r} V(F_i)) \setminus U$, and $W = \{w_{i,j} : i \in [r],\, j \in [k]\}$. Clearly, $\Delta(G) = k$, $S(G) = \emptyset$, $T(G) = U \cup W$ (as $k \geq 3$), and G is connected. Since U' is a Δ-reducing set of G, $\lambda(G) \leq rk$. Now, for every $i \in [r]$ and $j \in [k]$, each Δ-reducing set of G contains at least one member of $N_G[w_{i,j}] = V(H_{i,j}) \cup \{u_{i,j}\}$. Thus, $\lambda(G) = rk = |S(G)| + \frac{k}{k+1}|T(G)|$. $\square$

4 Proof of Theorem 2 and of Theorem 6

In this section, we prove Theorems 2 and 6.

Let mod* be the usual modulo operation with the exception that, for every $a \geq 1$ and $b \geq 0$, $ba \bmod^* a$ is a instead of 0.

For $0 \leq k < n$ such that kn is even (thus, n is even if k is odd), let $B_{n,k}$ be the graph defined by $V(B_{n,k}) = [n]$ and

$$E(B_{n,k}) = \begin{cases} \bigcup_{i=1}^{n} \left\{\{i, (i+j) \bmod^* n\} : j \in \left[\frac{k}{2}\right]\right\} & \text{if } k \text{ is even,} \\ \left\{\{i, i+\frac{n}{2}\} : i \in \left[\frac{n}{2}\right]\right\} \cup \bigcup_{i=1}^{n} \left\{\{i, (i+j) \bmod^* n\} : j \in \left[\frac{k-1}{2}\right]\right\} & \text{if } k \text{ is odd.} \end{cases}$$

Note that, for each $i \in [n]$, $N_{B_{n,k}}(i) = \{(i+j) \bmod^* n : j \in [k/2]\} \cup \{(i-j) \bmod^* n : j \in [k/2]\}$ if k is even, and $N_{B_{n,k}}(i) = \{(i+j) \bmod^* n : j \in [(k-1)/2]\} \cup \{(i-j) \bmod^* n : j \in [(k-1)/2]\} \cup \{(i+n/2) \bmod^* n\}$ if k is odd. Thus, $B_{n,k}$ is a k-regular n-vertex graph. Also, $B_{n,k}$ is connected if $k \geq 2$.

If k is even and $n \geq 3$, then $B_{n,k}$ is the $\frac{k}{2}$*th power* of the cycle C_n (the graph with vertex set $V(C_n)$ and where, for every two distinct vertices v and w, v and w are neighbours if and only if the distance between them in C_n is at most $\frac{k}{2}$). If k is odd and $n \geq 3$, then $B_{n,k}$ is obtained by adding the edges $\{1, 1+\frac{n}{2}\}, \dots, \{\frac{n}{2}, n\}$ to the $\frac{k-1}{2}$th power of C_n.

Construction 1 For any graph G and any even integer $p > \Delta(G) - \delta(G)$, we construct a $\Delta(G)$-regular graph $G \otimes p$ as follows. Let $k = \Delta(G)$ and $\ell = \delta(G)$. We take the union of p pairwise vertex-disjoint copies $G_1, \dots, G_p$ of G, and, for each vertex v of G with $d_G(v) < k$, we add the edges of a copy of $B_{p,k-d_G(v)}$ with vertex set $\{x : \text{for some } i \in [p],$ x is the vertex of G_i corresponding to $v\}$. More precisely, let $v_1, \dots, v_n$ be the distinct vertices of G, where $d_G(v_i) \leq d_G(v_{i+1})$ for each $i \in [n-1]$. For each $i \in [p]$, let $v_{i,j} = (i, v_j)$ for each $j \in [n]$, and let $G_i = (\{v_{i,j} : j \in [n]\}, \{v_{i,j}v_{i,j'} : j, j' \in [n],\, v_jv_{j'} \in E(G)\})$. Thus, $G_1, \dots, G_p$ are pairwise vertex-disjoint copies of G. Let $m = \min\{j \in [n] : d_G(v_j) = k\}$. For each $j \in [m-1]$, let $X_j = \{v_{i,j} : i \in [p]\}$ and $k_j = k - d_G(v_j)$. Since $\ell = d_G(v_1) \leq d_G(v_j)$, $k_j \leq k - \ell < p$. Let H_j be the copy of B_{p,k_j} with $V(H_j) = X_j$ and $E(H_j) = \{v_{i,j}v_{i',j} : i, i' \in [p],\, \{i, i'\} \in E(B_{p,k_j})\}$. We denote by $G \otimes p$ the graph with vertex set $\bigcup_{i=1}^{p} V(G_i)$ and edge set $\left(\bigcup_{i=1}^{p} E(G_i)\right) \cup \left(\bigcup_{j \in [m-1]} E(H_j)\right)$. We call $G \otimes p$ the *(G,p)-regularization graph.*

Lemma 1 *If G is an n-vertex graph with maximum degree k and minimum degree ℓ, then, for any even integer $p > k - \ell$, $G \otimes p$ is a k-regular pn-vertex graph and*

$$\lambda(G) \leq \frac{\gamma(G \otimes p)}{p}.$$

Proof. It is immediate that $|V(G \otimes p)| = pn$. For $i \in [p]$ and $j \in [m-1]$, we have $d_{G\otimes p}(v_{i,j}) = d_{G_i}(v_{i,j}) + d_{H_j}(v_{i,j}) = d_G(v_j) + k_j = k$. For $i \in [p]$ and $j \in [n]\backslash[m-1]$, we have $d_{G\otimes p}(v_{i,j}) = d_{G_i}(v_{i,j}) = d_G(v_j) = k$. Thus, $G \otimes p$ is k-regular.

Let D be a smallest dominating set of $G \otimes p$. Then, for each $v \in V(G \otimes p)$, there exists some $w_v \in D$ such that $v \in N_{G\otimes p}[w_v]$, and hence $w_v \in N_{G\otimes p}[v]$. For each $i \in [p]$, let $D_i = D \cap V(G_i)$. Then, $D = \bigcup_{i=1}^{p} D_i$ and $|D| = \sum_{i=1}^{p} |D_i|$. Thus, there exists some $r \in [p]$ such that $|D_r| \le |D|/p$ (as otherwise we obtain $\sum_{i=1}^{p} |D_i| > \sum_{i=1}^{p} |D|/p = |D|$, a contradiction). Now, for $i \in [p]$ and $v \in M(G_i)$, $N_{G\otimes p}[v] = N_{G_i}[v]$ (as $M(G_i) = \{v_{i,j} \colon j \in [n]\backslash[m-1]\}$), so $w_v \in N_{G_i}[v]$ and $w_v \in D_i$. Thus, for $i \in [p]$, D_i is a Δ-reducing set of G_i. Since G_r is a copy of G, we have $\lambda(G) = \lambda(G_r) \le |D_r| \le |D|/p = \gamma(G \otimes p)/p$. □

Lemma 2 *If G is a connected graph, $k = \Delta(G)$, $\ell = \delta(G)$, $p > k - \ell$, p is even, and either $\ell \le k-2$, or $\ell = k-1$ and $p = 2$, then $G \otimes p$ is connected.*

Proof. Recall that $d_G(v_1) \le \cdots \le d_G(v_n)$, so $d_G(v_1) = \ell$. If $\ell \le k-2$, then $k_1 \ge 2$, so H_1 is connected. If $\ell = k-1$, then $k_1 = 1$. Thus, if $\ell = k-1$ and $p = 2$, then H_1 is connected. Since each of $G_1, \dots, G_p$ is connected and contains a vertex of H_1, it follows that $G \otimes p$ is connected. □

Proof of Theorem 2. Since G is non-regular, $\ell \le k-1$. Let $p > k - \ell$ such that p is even. By Lemma 1, $G \otimes p$ is a k-regular pn-vertex graph and $\lambda(G) \le \gamma(G \otimes p)/p$.

Suppose that either $\ell \le k-2$, or $\ell = k-1$ and $p = 2$ $(= 2\left\lceil \frac{k-\ell+1}{2} \right\rceil)$. By Lemma 2, $G \otimes p$ is connected, so $\gamma(G \otimes p) \le \gamma_{\rm cr}(pn,k)$. Thus, $\lambda(G) \le \gamma_{\rm cr}(pn,k)/p = f(pn,k)n$. This settles (i) and part of (ii).

Now suppose that $\ell = k-1$, $p \ne 2$, and $\sum_{v\in V(G)} d_G(v) \ne kn-1$. Let $m' = m-1$. We have $m' \ge 1$ (as $d_G(v_1) = \ell = k-1$) and $kn - 1 \ne \sum_{v\in V(G)} d_G(v) = \sum_{j=1}^{m'} d_G(v_j) + \sum_{j=m'+1}^{n} d_G(v_j) = m'(k-1) + (n-m')k = kn - m'$. Thus, $m' \ge 2$, and hence $d_G(v_1) = d_G(v_2) = k-1$. Let R be the graph with vertex set $V(G \otimes p)$ and edge set $(E(G\otimes p)\backslash(E(H_1) \cup E(H_2))) \cup \{v_{i,1}v_{i+1,2} \colon i \in [p-1]\} \cup \{v_{p,1}v_{1,2}\}$. Then, R is a k-regular pn-vertex graph. Also, it is easy to see that, since G is connected, R is connected. Thus, $\gamma(R) \le \gamma_{\rm cr}(pn,k)$. By the argument in the proof of Lemma 1, $\lambda(G) \le \gamma(R)/p$. Therefore, we have $\lambda(G) \le \gamma_{\rm cr}(pn,k)/p = f(pn,k)n$. This settles the remaining part of (ii). □

Proof of Theorem 6. By Lemma 1, $G \otimes p$ is a k-regular pn-vertex graph and $\lambda(G) \le \gamma(G \otimes p)/p \le \gamma_{\rm r}(pn,k)/p = g(pn,k)n$. □

5 Problems, conjectures, and further results

In this section, we pose a number of problems and conjectures motivated by Theorems 2 and 6. Many domination problems arise naturally. We also establish a few additional results, mainly Theorem 8.

For $2 \le k < n$, we define

$$\lambda_{\rm c}(n,k) = \max\{\lambda(G) \colon G \text{ is a connected graph with } V(G) = [n] \text{ and } \Delta(G) = k\},$$

$$\lambda(n,k) = \max\{\lambda(G) \colon G \text{ is a graph with } V(G) = [n] \text{ and } \Delta(G) = k\}.$$

The graph $([n], \{\{1, i\} \colon i \in [k+1] \backslash \{1\}\} \cup \{\{i-1, i\} \colon i \in [n] \backslash [k+1]\})$ is a connected n-vertex graph with maximum degree k (so $\lambda_{\rm c}(n,k)$ exists).

Recall from Section 1 that, for $2 \leq k < n$, connected k-regular n-vertex graphs exist if and only if kn is even. For convenience, if kn is odd, then we take $\gamma_{\rm cr}(n,k) = \gamma_{\rm r}(n,k) = 0$.

The following is our ultimate and most direct question regarding the values considered here.

Problem 1 *What is the value of (i) $\lambda_{\rm c}(n,k)$? (ii) $\lambda(n,k)$? (iii) $\gamma_{\rm cr}(n,k)$? (iv) $\gamma_{\rm r}(n,k)$?*

However, it is understood that each of (i)–(iv) is a difficult problem.

As mentioned in Section 1, we ask whether Theorem 2 (ii) still holds without the degree sum condition, and we conjecture an affirmative answer.

Problem 2 *Does the inequality $\lambda(G) \leq f(pn,k)n$ in Theorem 2 (ii) still hold when $\sum_{v \in V(G)} d_G(v) = kn - 1$?*

Conjecture 1 *If G is a connected n-vertex graph, $k = \Delta(G) \geq 2$, $\ell = \delta(G)$, and $\sum_{v \in V(G)} d_G(v) = kn - 1$, then $\lambda(G) \leq f(pn,k)n$ for any even integer $p > k - \ell$.*

We note the following about Conjecture 1. A graph G with $\Delta(G) = k$ is k-regular if and only if $\sum_{v \in V(G)} d_G(v) = kn$. Thus, the graph in Conjecture 1 is closest to being k-regular (in terms of the degree sum). Indeed, let G be a non-regular n-vertex graph with $\Delta(G) = k$. Then, $d_G(u) \leq k-1$ for some $u \in V(G)$, so $\sum_{v \in V(G)} d_G(v) \leq k-1+(n-1)k = kn-1$. Moreover, $\sum_{v \in V(G)} d_G(v) = kn-1$ (as in Conjecture 1) if and only if $d_G(u) = k-1$ and $d_G(v) = k$ for each $v \in V(G) \backslash \{u\}$. Since $\sum_{v \in V(G)} d_G(v) = 2|E(G)|$ (by the handshaking lemma), if $\sum_{v \in V(G)} d_G(v) = kn-1$, then k and n are odd.

Problem 3 *(i) For each of Theorem 2 (i), Theorem 2 (ii), and Theorem 6, for which integers k is the bound in the theorem infinitely attainable with k fixed?*

(ii) For each of Theorem 2 (ii) and Theorem 6, for which pairs (k,p) is the bound in the theorem infinitely attainable with k and p fixed?

Remark 1 (iv) and Remark 2 (ii) answer Problem 3 (i) for $k = 2$; they give us that, for $k = 2$, Theorem 2 (i) is infinitely attainable, Theorem 2 (ii) is infinitely attainable for any fixed even $p > 0$, and Theorem 6 is infinitely attainable for $p = 2$. By Remark 2 (iii), Theorem 6 is infinitely attainable for $k = 3$ and any fixed even $p > 0$.

Problem 4 *What is the smallest integer $q \geq n$ such that*

(i) $\lambda_{\rm c}(n,k) \leq \gamma_{\rm cr}(q,k)$?

(ii) $\lambda_{\rm c}(n,k) \leq f(q,k)n$?

(iii) $\lambda(n,k) \leq \gamma_{\rm r}(q,k)$?

(iv) $\lambda(n,k) \leq g(q,k)n$?

Conjecture 2 *For $2 \leq k < n$, $\lambda_{\rm c}(n,k) = \gamma_{\rm cr}(n,k)$ or $\lambda_{\rm c}(n,k) \leq \gamma_{\rm cr}(n+1,k)$.*

Conjecture 3 *For $1 \leq k < n$, $\lambda(n,k) = \gamma_{\rm r}(n,k)$ or $\lambda(n,k) \leq \gamma_{\rm r}(n+1,k)$.*

Next, we instead ask for an integer $q^* \geq n$ that gives the closest bound.

Problem 5 *For each of the four parts below, determine an integer $q^* \geq n$ for which the part holds:*

(i) $\lambda_{\rm c}(n,k) \leq \gamma_{\rm cr}(q^*,k) = \min\{\gamma_{\rm cr}(q,k)\colon q \geq n,\ \lambda_{\rm c}(n,k) \leq \gamma_{\rm cr}(q,k)\}$.

(ii) $\lambda_{\rm c}(n,k) \leq f(q^*,k)n = \min\{f(q,k)n\colon q \geq n,\ \lambda_{\rm c}(n,k) \leq f(q,k)n\}$.

(iii) $\lambda(n,k) \leq \gamma_{\rm r}(q^*,k) = \min\{\gamma_{\rm r}(q,k)\colon q \geq n,\ \lambda(n,k) \leq \gamma_{\rm r}(q,k)\}$.

(iv) $\lambda(n,k) \leq g(q^*,k)n = \min\{g(q,k)n\colon q \geq n,\ \lambda(n,k) \leq g(q,k)n\}$.

For each of the parts (i)–(iv) of Problem 4, we also ask if the integer q is a solution for the corresponding part in Problem 5.

We have $f(n,k) \leq g(n,k)$. For which values of n and k is $f(n,k) = g(n,k)$? This question is equivalent to the following.

Problem 6 *For which values of n and k is $\gamma_{\rm cr}(n,k) = \gamma_{\rm r}(n,k)$?*

We conjecture that $\{n \in \mathbb{N}\colon \gamma_{\rm cr}(n,k) = \gamma_{\rm r}(n,k) \neq 0\}$ is a finite set.

Conjecture 4 *For $k \geq 2$, there are finitely many integers n for which $\gamma_{\rm cr}(n,k) = \gamma_{\rm r}(n,k) \neq 0$.*

Proposition 1 *Conjecture 4 is true if $k = 2$ or $k = 3$.*

Proof. Consider $k = 2$ and $n \geq 18$. Let $t = \lfloor \frac{n-3}{4} \rfloor$. Then, $t \geq \frac{n-6}{4}$. Let $G_1, \dots, G_t, G_{t+1}$ be pairwise vertex-disjoint graphs, where $G_1, \dots, G_t$ are copies of C_4, and G_{t+1} is a copy of C_{n-4t}. Let G be the union of $G_1, \dots, G_t, G_{t+1}$. Since G is a 2-regular n-vertex graph, $\gamma_{\rm r}(n,2) \geq \gamma(G) \geq 2t+1 \geq \frac{n-6}{2}+1 \geq \frac{n}{3}+1 > \lceil \frac{n}{3} \rceil = \gamma(C_n) = \gamma_{\rm cr}(n,2)$ (as every 2-regular graph is a cycle).

Now consider $k = 3$ and any even $n \geq 176$. Let $t = \lfloor \frac{n-4}{8} \rfloor$. Then, $t \geq \frac{n-11}{8}$. Let $G_1, \dots, G_t, G_{t+1}$ be pairwise vertex-disjoint graphs, where $G_1, \dots, G_t$ are copies of the graph C_8' in Section 2, and G_{t+1} is a copy of $B_{n-8t,3}$ ($n - 8t$ is even as n is even). Let G be the union of $G_1, \dots, G_t, G_{t+1}$. Since G is a 3-regular n-vertex graph, $\gamma_{\rm r}(n,3) \geq \lambda(G) \geq 3t+1 \geq \frac{3(n-11)}{8}+1 > \frac{5}{14}n \geq \gamma_{\rm cr}(n,3)$ by the KS bound mentioned in Section 2. $\Box$

With the notation in Section 2, we have ${\lambda_{\rm c}}'(n,k) = \frac{\lambda_{\rm c}(n,k)}{n}$ and $\lambda'(n,k) = \frac{\lambda(n,k)}{n}$.

Problem 7 *Given k, how does each of the following functions behave as n increases? (i) $\lambda_{\rm c}(n,k)$, (ii) $\lambda(n,k)$, (iii) $\gamma_{\rm cr}(n,k)$, (iv) $\gamma_{\rm r}(n,k)$, (v) ${\lambda_{\rm c}}'(n,k)$, (vi) $\lambda'(n,k)$, (vii) $f(n,k)$, (viii) $g(n,k)$.*

We have the following answers. By Theorem 4, the values $\lambda_{\rm c}(n,k)$, $\lambda(n,k)$, $\gamma_{\rm cr}(n,k)$, and $\gamma_{\rm r}(n,k)$ are bounded above by the bound in the theorem (recall that $\gamma(G) = \lambda(G)$ if G is regular) for $k \geq 3$, and by $\frac{n}{2}$ for $k = 2$. As explained below, these values are bounded below by $\frac{n}{k+1}$ if kn is even, and $\lambda_{\rm c}(n,k)$ and $\lambda(n,k)$ are bounded below by $\frac{n-1}{k+1}$ if kn is odd (recall that $\gamma_{\rm cr}(n,k) = \gamma_{\rm r}(n,k) = 0$ if kn is odd); thus, the values grow to infinity as n grows. For (i) and (ii) (of Problem 7), we have a more precise answer. For any $k \geq 2$, $\lambda_{\rm c}(n,k)$ and $\lambda(n,k)$ are increasing functions of n. This is given by parts (i) and (ii) of our next result.

Theorem 8 *For $2 \leq k < n$,*

(i) $\lambda_{\rm c}(n+1,k) \geq \lambda_{\rm c}(n,k)$,

(ii) $\lambda(n+1,k) \geq \lambda(n,k)$,

(iii) $\lambda_{\rm c}(n+k+1,k) \geq \lambda_{\rm c}(n,k)+1$,

(iv) $\lambda(n+k+1,k) \geq \lambda(n,k)+1$.

Moreover, if kn is even, then

(v) $\gamma_{\rm cr}(n+k+1,k) \geq \gamma_{\rm cr}(n,k)+1$,

(vi) $\gamma_{\rm r}(n+k+1,k) \geq \gamma_{\rm r}(n,k)+1$.

In order to prove Theorem 8 (iii)–(vi), we establish the following.

Lemma 3 *If G is an n-vertex graph with $\Delta(G) = k \geq 2$, G' is a copy of K_{k+1} with $V(G) \cap V(G') = \emptyset$, $vw \in E(G)$, $v'w' \in E(G')$, and*

$$H = (V(G) \cup V(G'), ((E(G) \cup E(G')) \backslash \{vw, v'w'\}) \cup \{vv', ww'\}),$$

then H is an $(n+k+1)$-vertex graph with $\Delta(H) = k$ and

$$\lambda(H) = \lambda(G) + 1.$$

Moreover, H is k-regular if G is k-regular, and H is connected if G is connected.

Proof. Since $V(G) \cap V(G') = \emptyset$, $|V(H)| = |V(G)| + |V(G')| = n+k+1$. We have $d_H(v) = d_G(v)$, $d_H(w) = d_G(w)$, $d_H(v') = d_{G'}(v')$, and $d_H(w') = d_{G'}(w')$. Thus, $\Delta(H) = \Delta(G)$, and if G is k-regular, then H is k-regular. Clearly, H is connected if G is connected. It remains to show that $\lambda(H) = \lambda(G)+1$, which is equivalent to $\lambda(G)+1 \leq \lambda(H) \leq \lambda(G)+1$. We have $V(G') \subseteq M(H)$ and, since $k+1 \geq 3$, $V(G') \backslash \{v', w'\} \neq \emptyset$.

Let X be a smallest Δ-reducing set of G. If $v \in X$, then $X \cup \{w'\}$ is a Δ-reducing set of H. If $w \in X$, then $X \cup \{v'\}$ is a Δ-reducing set of H. If $v, w \notin X$, then, for any $y \in V(G') \backslash \{v', w'\}$, $X \cup \{y\}$ is a Δ-reducing set of H. Thus, $\lambda(H) \leq |X|+1 = \lambda(G)+1$.

Let D be a smallest Δ-reducing set of H. Let $D_G = D \cap V(G)$ and $D_{G'} = D \cap V(G')$. Then, $\lambda(H) = |D| = |D_G| + |D_{G'}|$ and, since $\emptyset \neq V(G') \backslash \{v', w'\} \subseteq M(H)$, $D_{G'} \neq \emptyset$. If D_G is a Δ-reducing set of G, then $\lambda(G) \leq |D_G|$, so $\lambda(H) \geq \lambda(G)+1$. Suppose that D_G is not a Δ-reducing set of G. Then, $D \cap \{v, w\} = \emptyset$, $x \in D$ for some $x \in \{v', w'\}$, and $D_G \cup \{v\}$ is a Δ-reducing set of G. Since $D \cap \{v, w\} = \emptyset$, $v'w' \notin E(H)$, and D dominates $\{v', w'\} \backslash \{x\}$

in H, we have $D_{G'}\backslash\{x\} \neq \emptyset$. Therefore, $\lambda(H) \geq |D_G|+2 = |D_G \cup \{v\}|+1 \geq \lambda(G)+1$. $\Box$

Proof of Theorem 8. Let G be an arbitrary n-vertex graph with $\Delta(G) = k$. Let $x \in \mathbb{N}$ such that $x \notin V(G)$, and let $v, w \in V(G)$ such that $vw \in E(G)$. Let $H = (V(G) \cup \{x\}, (E(G)\backslash\{vw\}) \cup \{vx, xw\})$. Then, H is an $(n+1)$-vertex graph with $\Delta(H) = k$. Clearly, H is connected if G is connected. Let D be a smallest Δ-reducing set of H. If $x \notin D$, then D is a Δ-reducing set of G. If $x \in D$, then $(D\backslash\{x\}) \cup \{v\}$ is a Δ-reducing set of G. Thus, $\lambda(G) \leq |D| = \lambda(H)$. By taking G with $\lambda(G) = \lambda(n,k)$, we obtain $\lambda(n+1,k) \geq \lambda(n,k)$, and, by Lemma 3, we also obtain $\lambda(n+k+1,k) \geq \lambda(n,k)+1$. By taking a connected graph G with $\lambda(G) = \lambda_{\rm c}(n,k)$, we obtain $\lambda_{\rm c}(n+1,k) \geq \lambda_{\rm c}(n,k)$, and, by Lemma 3, we also obtain $\lambda_{\rm c}(n+k+1,k) \geq \lambda_{\rm c}(n,k)+1$. Suppose that kn is even. By taking a k-regular graph G with $\gamma(G) = \gamma_{\rm r}(n,k)$, we obtain $\gamma_{\rm r}(n+k+1,k) \geq \gamma_{\rm r}(n,k)+1$ from Lemma 3 (again recall that $\gamma(G) = \lambda(G)$ if G is regular). By taking a connected k-regular graph G with $\gamma(G) = \gamma_{\rm cr}(n,k)$, we obtain $\gamma_{\rm cr}(n+k+1,k) \geq \gamma_{\rm cr}(n,k)+1$ from Lemma 3. $\Box$

We now prove the statement above regarding the lower bounds for the values in Problem 7 (i)–(iv). As observed in [1], for any graph G,

$$\lambda(G) \geq \frac{|M(G)|}{\Delta(G)+1}. \tag{14}$$

Let $2 \leq k < n$. Suppose that kn is even. Since $B_{n,k}$ is k-regular, we have $M(B_{n,k}) = V(B_{n,k}) = [n]$, so $\lambda(B_{n,k}) \geq \frac{n}{k+1}$ by (14). Now $\lambda(B_{n,k}) \leq \gamma_{\rm cr}(n,k)$ as $B_{n,k}$ is connected. Also, we clearly have $\gamma_{\rm cr}(n,k) \leq \gamma_{\rm r}(n,k)$ and $\gamma_{\rm cr}(n,k) \leq \lambda_{\rm c}(n,k) \leq \lambda(n,k)$. Therefore, $\frac{n}{k+1}$ is a lower bound for each of $\gamma_{\rm cr}(n,k)$, $\gamma_{\rm r}(n,k)$, $\lambda_{\rm c}(n,k)$, and $\lambda(n,k)$. Now suppose that kn is odd. Then, k and n are odd. Thus, $n-1$ is even, and, since $2 \leq k < n$, we actually have $3 \leq k \leq n-2$. Let G be the graph with $V(G) = [n] = V(B_{n-1,k}) \cup \{n\}$ and $E(G) = (E(B_{n-1},k)\backslash\{\{n-1,1\}\}) \cup \{\{n-1,n\},\{n,1\}\}$. Since $\Delta(G) = k$ and $M(G) = [n-1]$, $\lambda(G) \geq \frac{n-1}{k+1}$ by (14). Since G is connected, $\lambda(G) \leq \lambda_{\rm c}(n,k)$. Thus, we have $\frac{n-1}{k+1} \leq \lambda_{\rm c}(n,k) \leq \lambda(n,k)$.

In view of Theorem 8 (and Problem 7), we ask if, by fixing k and excluding the cases where kn is odd, we also have that $\gamma_{\rm cr}(n,k)$ and $\gamma_{\rm r}(n,k)$ are increasing functions of n. More precisely, we pose the following problem.

Problem 8 *(i) Is $\gamma_{\rm cr}(n+1,k) \geq \gamma_{\rm cr}(n,k)$ for $2 \leq k < n$ with k even?*

(ii) Is $\gamma_{\rm r}(n+1,k) \geq \gamma_{\rm r}(n,k)$ for $2 \leq k < n$ with k even?

(iii) Is $\gamma_{\rm cr}(n+2,k) \geq \gamma_{\rm cr}(n,k)$ for $2 \leq k < n$ with kn even?

(iv) Is $\gamma_{\rm r}(n+2,k) \geq \gamma_{\rm r}(n,k)$ for $2 \leq k < n$ with kn even?

For each of (i)–(iv), we conjecture an affirmative answer.

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References

[1] P. Borg and K. Fenech, Reducing the maximum degree of a graph by deleting vertices, Australas. J. Combin. 69(1) (2017), 29–40.

[2] P. Borg and K. Fenech, Reducing the maximum degree of a graph by deleting vertices: the extremal cases, Theory Appl. Graphs 5(2) (2018), article 5.

[3] E.J. Cockayne, Domination of undirected graphs – A survey, Lecture Notes in Mathematics, Volume 642, Springer, 1978, 141–147.

[4] M. Chellali, O. Favaron, A. Hansberg and L. Volkmann, k-Domination and k-Independence in Graphs: A Survey, Graphs Combin. 28 (2012), 1–55.

[5] E.J. Cockayne and S.T. Hedetniemi, Towards a theory of domination in graphs, Networks 7 (1977), 247–261.

[6] W.J. Desormeaux and M.A. Henning, Paired domination in graphs: a survey and recent results, Util. Math. 94 (2014), 101–166.

[7] K. Fenech, Graph theory results of independence, domination, covering and Turán type, Ph.D. Thesis, University of Malta, 2019.

[8] W. Goddard and M.A. Henning, Independent domination in graphs: A survey and recent results, Discrete Math. 313 (2013), 839–854.

[9] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, Inc., New York, 1998.

[10] T.W. Haynes, S.T. Hedetniemi and P.J. Slater (Editors), Domination in Graphs: Advanced Topics, Marcel Dekker, Inc., New York, 1998.

[11] S.T. Hedetniemi and R.C. Laskar (Editors), Topics on Domination, Discrete Math., Volume 86, 1990.

[12] S.T. Hedetniemi and R.C. Laskar, Bibliography on domination in graphs and some basic definitions of domination parameters, Discrete Math. 86 (1990), 257–277.

[13] M.A. Henning, A survey of selected recent results on total domination in graphs, Discrete Math. 309 (2009), 32–63.

[14] M.A. Henning and A. Yeo, Total domination in graphs, Springer Monographs in Mathematics, Springer, New York, 2013.

[15] C. Jianxiang, S. Minyong, M.Y. Sohn and Y. Xudong, Domination in graphs with minimum degree six, J. Appl. Math. & Informatics 26 (2008), 1085–1100.

[16] A. Kelmans, Counterexamples to the cubic graph domination conjecture, arXiv:math/0607512 [math.CO].

[17] A.V. Kostochka and B.Y. Stodolsky, On domination in connected cubic graphs, Discrete Math. 304 (2005), 45–50.

[18] A.V. Kostochka and B.Y. Stodolsky, An upper bound on the domination number of n-vertex connected cubic graphs, Disc. Math. 309 (2009), 1142–1162.

[19] A.V. Kostochka and C. Stocker, A new bound on the domination number of connected cubic graphs, Sib. Èlektron. Mat. Izv. 6 (2009), 465–504.

[20] W. McCuaig and B. Shepherd, Domination in graphs with minimum degree two, J. Graph Theory 13 (1989), 749–762.

[21] O. Ore, Theory of graphs, American Mathematical Society Colloquium Publications, Volume 38, American Mathematical Society, Providence, R.I., 1962.

[22] B. Reed, Paths, stars, and the number three, Combin. Probab. Comput. 5 (1996), 277–295.

[23] M.Y. Sohn and Y. Xudong, Domination in graphs of minimum degree four, J. Korean Math. Soc. 46 (2009), 759–773.

[24] H.-M. Xing, L. Sun and X.-G Chen, Domination in graphs of minimum degree five, Graphs Combin. 22 (2006), 127–143.

[25] W. Yu, C. Zheng, W. Cheng, C.C. Aggarwal, D. Song, B. Zong, H. Chen and W. Wang, Learning deep network representations with adversarially regularized autoencoders, Proceedings of the ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, ACM, New York, 2018, pp. 2663–2671.